Recall : Given a orientable regular surface S.

Shape operator $S_p : T_pS \to T_pS$ where

S_p is self-adjoint wrt inner product $g = \langle \, , \, \rangle|_{T_pS}$

i.e. $\langle S_p(v), w \rangle = \langle v, S_p(w) \rangle$, $\forall\, v, w \in T_pS$, $p \in S$.

On T_pS, let $\beta = \{e_1, e_2\}$ be O.N. base.

i.e. $[g_p]_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in GL(2, \mathbb{R})$.

then $S_p = S_p^* \Rightarrow [S_p]_\beta = [S_p]_\beta^T$ (symmetric)

linear algebra $\Rightarrow \exists\, Q \in SO(2)$ s.t.

$$QAQ^T = \text{diagonal where } A = [S_p]_\beta.$$

$$= \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Equivalently, $\exists\, v_1, v_2$ s.t. $\begin{cases} S_p(v_1) = \lambda_1 v_1 \\ S_p(v_2) = \lambda_2 v_2. \end{cases}$

$\Downarrow$

The second fundamental form $\mathrm{II}(\cdot,\cdot) = \langle S_p(\cdot), \cdot \rangle$

is in form of

$$\mathrm{II}(u, u) = \lambda_1 x^2 + \lambda_2 y^2 \quad \text{where} \quad u = x v_1 + y v_2.$$

in matrix form:

$$= [x \quad y] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [\lambda_1 x^2 + \lambda_2 y^2] \#$$

$$\therefore \quad \lambda_1 = \underbrace{\max}_{} \{ \mathrm{II}(v,v) \mid |v| = 1 \}$$

$$\lambda_2 = \underbrace{\min}_{} \{ \mathrm{II}(v,v) \mid |v| = 1 \}$$

depends on choice of orientations.

Recall: $K = \lambda_1 \lambda_2 \triangleq$ Gaussian curvature ← indep. of orientation

$H = \frac{\lambda_1 + \lambda_2}{2} \triangleq$ mean curvature ← dep. on orientation.

Q: How to compute K and H using $X: \mathcal{U} \to S$??

If $\alpha = \{u_1, u_2\}$, the eigen-basis is found,

then $[S_p]_\alpha = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$ ← easy

difficult to be found !! $= D$

$D = Q A Q^T$ where $A = [S_p]_\beta$, β = o.n. basis

$$\Rightarrow \begin{cases} \lambda_1 + \lambda_2 = \operatorname{tr}(D) = \operatorname{tr}(A) \\ \lambda_1 \lambda_2 = \det(D) = \det(A) \end{cases}$$

* Both tr, det are invariant under $GL(n,\mathbb{R})$.

i.e. $\begin{cases} \mathrm{tr}(QAQ^{-1}) = \mathrm{tr}(A) \\ \det(QAQ^{-1}) = \det(A). \end{cases}$

$A = [S_p]_\beta$ where β = eigen-basis.

wrt standard basis $\alpha = \{X_u, X_v\}$, we write

$[S_p]_\alpha = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ where $S_p(X_i) = a_{ij} X_j$

if we denote $\begin{cases} X_1 = X_u \\ X_2 = X_v. \end{cases}$

$\Rightarrow \begin{cases} \frac{1}{2}\mathrm{tr}[S_p]_\alpha = H \\ \det [S_p]_\alpha = K \end{cases}$ (since $[S_p]_\alpha = Q[S_p]_\beta Q^{-1}$ for some Q)

Find $[S_p]_\alpha$:

$$S_p(X_i) = a_{ij} X_j$$

$$\Rightarrow \langle S_p(X_i), X_j \rangle = a_{il}\, g_{lj}$$

$$\therefore \begin{bmatrix} \mathrm{II}(X_1, X_1) & \mathrm{II}(X_1, X_2) \\ \mathrm{II}(X_1, X_2) & \mathrm{II}(X_2, X_2) \end{bmatrix} = [S_p]_\alpha [g]_X$$

Hence, ~~prop~~:

$$\begin{cases} K(p) = \dfrac{eg - f^2}{EG - F^2} \\ H(p) = \dfrac{1}{2} \dfrac{eG - 2fF + gE}{EG - F^2} \end{cases}$$ where

$$[\mathrm{II}]_X = \begin{bmatrix} e & f \\ f & g \end{bmatrix}, \quad \begin{cases} e = \mathrm{II}(X_1, X_1) \\ f = \mathrm{II}(X_1, X_2) = \mathrm{II}(X_2, X_1) \\ g = \mathrm{II}(X_2, X_2) \end{cases}$$

Moreover,

$$\begin{aligned} \mathrm{II}(X_i, X_j) &= \langle S_p(X_i), X_j \rangle = \langle S_p(X_j), X_i \rangle \\ &= -\langle dN_p(X_i), X_j \rangle \\ &= -\langle N_i, X_j \rangle \quad \left(N_i = \tfrac{\partial}{\partial x_i} N \right) \\ &= \langle N, X_{ij} \rangle. \end{aligned}$$

where $N = \dfrac{X_u \times X_v}{\| X_u \times X_v \|}$. and Lemma.

$$\begin{cases} e = \dfrac{\det(X_u, X_v, X_{uv})}{\sqrt{EG - F^2}} \\[2ex] f = \dfrac{\det(X_u, X_v, X_{uv})}{\sqrt{EG - F^2}} \\[2ex] g = \dfrac{\det(X_u, X_v, X_{vv})}{\sqrt{EG - F^2}} \end{cases}$$

$\longleftarrow$ the triple product in calculus.

$\langle u \times v, w \rangle = \det(u, v, w)$.